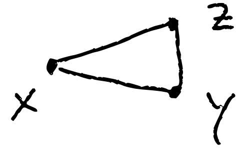
Lecture 2.

Def. ① A *metric space* (X,d) is a set X w/ a fcn $d: X\times X \to [0,\infty)$ s.t.

- $d(x,y)\geq 0$ and $d(x,y)=0 \iff x=y$.
- $d(x,y)=d(y,x)$.
- $d(x,z)\leq d(x,y)+d(y,x)$ (Δ-ineq.)



Ex. ① $(\mathbb{R}^n, d_E(x,y)=|x-y|)$ and $|x|^2=\sum_{k=1}^n x_k^2$. E.g. $(\mathbb{C}=\mathbb{R}^2, d_E)$.

② $(\mathbb{C}_\infty, d_\infty)$ d_∞ is Fubini-Study as defined last lecture (or C-I.6)

Prop 1. If (X,d) is metric space, $A\subseteq X$, then (A,d) is also metric space.

Pf. Obvious.

Def. ② If (X,d) is metric space, *open/closed balls* centered at x_0 of radius r are

$$B(x_0,r)=\{x: d(x,x_0)<r\}, \text{ (open ball)}$$

$\overline{B}(x_0, r) = \{x : d(x, x_0) \le r\}$ (closed ball)

(3) $U \subseteq X$ is <u>open</u> if $\forall x_0 \in U \ \exists \varepsilon > 0$ s.t. $B(x_0, \varepsilon) \subseteq U$. (or $U = \emptyset$).

(4) $F \subseteq X$ is <u>closed</u> if $X \setminus F$ is open.

<u>Prop 2</u> (i) If $U_1, \dots, U_n \subseteq X$ are open, then $\bigcap_{k=1}^{n} U_k$ is open.

(ii) If $\{U_\alpha\}_{\alpha \in A}$ are open, then (A any index set)

$$\bigcup_{\alpha \in A} U_\alpha \text{ is open.}$$

<u>Pf.</u> DIY. | $\Rightarrow$ similar Prop for closed sets |

interior

<u>Def.</u> (5) Given $A \subseteq X$, $\operatorname{int} A = \mathring{A} = \bigcup\{G : G \subseteq A \text{ open}\}$ $\left(= \bigcup\{B(x,r) : B(x,r) \subseteq A\}\right)$.

(6) Closure of A, $\overline{A} = \bigcap\{F : A \subseteq F \text{ closed}\}$.

(7) Boundary of A, $\partial A = \overline{A} \cap \overline{(X \setminus A)}$

Prop 3. (i) A open $\Leftrightarrow$ $A = \text{int } A$.

(ii) A closed $\Leftrightarrow$ $A = \overline{A}$.

(iii) $\text{int } A = X \setminus \overline{(X \setminus A)}$

(iv) $\overline{A} = X \setminus \text{int}(X \setminus A)$

(v) $\partial A = \overline{A} \setminus \text{int } A$.

+ more (see Conway).

Def. (8) $A \subseteq X$ is dense if $\overline{A} = X$.

Connectedness.

Def. (9) A metric space (X, d) is connected if $\emptyset$ and X are the only subsets that are both open and closed. (A subset $A \subseteq X$ is connected if (A, d) is connected as metric space.)

Prop 4. $X \subseteq \mathbb{R}$ is connected $\Leftrightarrow$ X is an interval.

Pf. Let's show $X = (a, b)$ is connected. (similar arguments for other intervals give $\Leftarrow$.)

Let $B \subseteq (a,b)$ be non-empty, open & closed. Must show $B = (a,b)$. Suppose not. Then $\exists c: a < c < b$ s.t. $c \notin B$. Then $B = B_1 \cup B_2$, $B_1 = B \cap (a,c)$, $B_2 = B \cap (c,b)$. B_1, B_2 are both open $\Rightarrow X \setminus B_1, X \setminus B_2$ are closed. Since $B_1 = B \cap (X \setminus B_2)$ and $B_2 = B \cap (X \setminus B_1)$, B_1, B_2 are also closed.

WLOG $B_1 \neq \emptyset$. Let $d = \sup B_1 \leq c$. Since B_1 is closed, $d \in B_1$ ($B(d,\varepsilon) \cap B_1 \neq \emptyset$ $\forall \varepsilon > 0$). Thus, $d < c$ since $c \notin B_1$. But $B(d,\varepsilon) \not\subseteq B_1$ for any ε since $B(d,\varepsilon)$ contains points $d < x < c$ and $d = \sup B_1$. This is a contradiction since B_1 is open. Thus, $B = (a,b)$ as desired.

$\Rightarrow$. HW next week.

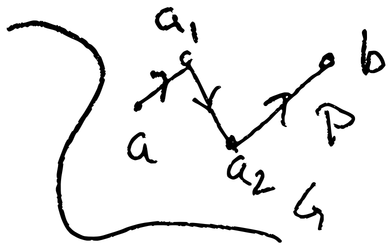
Thm Let $G \subseteq \mathbb{C}$ be open set.
G is connected $\Leftrightarrow$ $\forall a, b \in G$ $\exists$ polygonal path $P \subseteq G$ from a to b.

Rem. May take P to comprise only vertical and horizontal line segments.

Pf. $\Rightarrow$: Fix $a \in G$ and let $A = \{b \in G : \exists P \subseteq G$ connecting a to $b\}$. Since G is open $\exists$ $B(a, \varepsilon) \subseteq G$ and $B(a,\varepsilon) \subseteq A \neq \emptyset$. For the same reason, if $b \in A$, then $B(b, \delta) \subseteq G$ and hence $B(b,\delta) \subseteq A$, so A is open.

Now, we show A is closed. Let $b \in G$ be in $\overline{A}$. Since G open, $\exists \alpha > 0$ s.t. $B(b, \alpha) \subseteq G$. Since $b \in \overline{A}$, $\exists$ $b' \in A$ and $b' \in B(b, \alpha)$. Thus, $[b', b] \subseteq B(b, \alpha) \subseteq G$ and $b \in A \Rightarrow A$ is closed. But $A \neq \emptyset$, A is both open/closed, so G connected $\Rightarrow A = G$, which proves $\Rightarrow$.

$\Leftarrow$: Assume $\exists$ polygonal $P \subseteq G$ between any $a, b \in G$. Assume, to obtain contradiction, G is not connected. Then $\exists$ $\emptyset \neq A \subsetneq G$ both open and closed. Pick $a \in A$ and $b \in G \setminus A$ and a polygonal P



$$P = \bigcup_{j=1}^{n} [a_{j-1}, a_j]$$

w/ $a_0 = a$, $a_n = b$.

WLOG. $\exists$ $[a_{j-1}, a_j]$ s.t. $a_{j-1} \in A$, $a_j \notin A$.

Let $\gamma(t) = (1-t)a_{j-1} + t a_j$, $t \in [0,1]$. This parametrizes $[a_{j-1}, a_j]$. Let $T := \{t \in [0,1] : \gamma(t) \in A\}$. $\exists$ $[0, \varepsilon) \subseteq T$ as $a_{j-1} \in A$ and A open. Also, $\exists$ $(1-\delta, 1] \subseteq [0,1] \setminus T$ since $a_j \in G \setminus A$ and $G \setminus A$ open. $\Rightarrow$ $\emptyset \neq T \subsetneq [0,1]$. A similar argument ($A$ is open and $G \setminus A$ is open) shows T is open and closed. By Prop 4, this is a contradiction since $[0,1]$ is connected.